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A Lie Group Structure for Pseudodifferential Operators

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1. Introduction

In 1979 Adler gave in [3] a Lie algebraic explanation of the complete integrability of the Korteweg-de Vries (KdV) equation. The Lie algebra used there is the algebra of formal pseudodifferential operators on the circle, S^1. (Adler was a bit more general than this but we restrict ourselves to this special case.) A formal pseudodifferential operator on S^1 is a formal series $P(x,\xi)=\sum_{j=-\infty}^{m} a_j(x)\xi^j$ with $a_j(x)\in C^\infty(S^1)$. Composition is defined by the formula

$$P\circ Q(x,\xi)=\sum_{k\geqq 0}\frac{1}{k!}\partial_\xi^k P(x,\xi)\partial_x^k Q(x,\xi)\,.$$

The space of formal pseudodifferential operators, $\psi DO(S^1)$ is a topological Lie algebra with the bracket $[P,Q]=P\circ Q-Q\circ P$, and the topology given by the infinite product $C^\infty(S^1)\times C^\infty(S^1)\times\ldots$ (see also Iacob and Sternberg [11]). With this Lie algebra Adler shows that KdV gives an infinite dimensional analog of the complete integrability of certain systems on Lie algebras described by Kostant and Symes in [12, 18, 19].

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It is natural to ask what Lie group has $\psi DO(S^1)$ as its Lie algebra. Formally speaking it is easy to see that the group of invertible Fourier integral operators on S^1 has the pseudodifferential operators as Lie algebra. To make this statement rigorous one has to give a "Lie group" structure to the invertible Fourier integral operators. The purpose of this paper along with the forthcoming paper [2] is to give a Lie group structure to the group of invertible formal Fourier integral operators on a compact manifold M and show that the Lie algebra of this group is the Lie algebra of formal pseudodifferential operators on M. We describe below our strategy for doing this.

Consider the group $(\mathrm{FIO})_*$ of invertible classical Fourier integral operators on M, the group operation being operator multiplication. The canonical relation of a Fourier integral operator is a conic, Lagrangian submanifold of $(T^*M\backslash O)\times(T^*M\backslash O)$. If it is invertible, the canonical relation must be the graph of a canonical transformation η on $T^*M\backslash O$. Since graph $(\eta)\subset(T^*M\backslash O)\times(T^*M\backslash O)$ is assumed conic and Lagrangian η must preserve the canonical one-form θ on T^*M, i.e. $\eta^*\theta=\theta$. Let $\mathscr{D}_\theta(T^*M\backslash O)$ be the space of smooth diffeomorphisms of $T^*M\backslash O$ which preserve θ. Then there is a surjective map

$$\pi:(\mathrm{FIO})_*\to\mathscr{D}_\theta(T^*M\backslash O).$$

If we denote the identity symplectomorphism by e, we note that $\pi^{-1}(e)=(\psi DO)_*$, the space of invertible classical pseudodifferential operators. $(\psi DO)_*$ is also a group under operator multiplication and we have the exact sequence of groups

$$I\to(\psi DO)_*\xrightarrow{j}(\mathrm{FIO})_*\xrightarrow{\pi}\mathscr{D}_\theta(T^*M\backslash O)\to e, \tag{1.1}$$

where I is the identity operator and j is inclusion.

In Sect. 2 of this paper we describe the spaces of formal pseudodifferential operators and Fourier integral operators on a compact manifold M, i.e. those which have classical symbols and are defined modulo smoothing operators. For these spaces we still get the exact sequence (1.1). The technique we use to give a Lie group structure to the group in the middle of this sequence is first to give Lie group structures to both ends of the sequence and then to produce a local section of the fibration $\pi:(\mathrm{FIO})_*\to\mathscr{D}_\theta(T^*M\backslash O)$ satisfying compatibility conditions that will allow us to define charts on $(\mathrm{FIO})_*$.

The space $\mathscr{D}_\theta(T^*M\backslash O)$ can of course be given a Fréchet manifold structure on which multiplication is smooth; but as with the diffeomorphism group $\mathscr{D}(M)$ there is a much richer Lie group structure, namely the ILH (inverse limit Hilbert) structure described by Omori [15]. Briefly, an ILH-Lie group structure for a topological group G is a collection of topological groups $(G^s)_{s\geqq s_0}$ such that

1) G^s is a Hilbert manifold
2) $G^{s+1}\subset G^s$, densely and continuously
3) $G=\cap G^s$ with the inverse limit topology
4) $M_{s,t,r}:G^s\times G^t\to G^r$ defined by $(g_1,g_2)\mapsto g_1g_2$ is C^{s-r} for $s,t\geqq r$.
5) Technical assumptions.

For $G=\mathscr{D}_\theta(T^*M\backslash O)$ the groups G^s are the spaces $\mathscr{D}^s_\theta(T^*M\backslash O)$, $s>\dim M+1$, of H^s-diffeomorphisms of $T^*M\backslash O$ which preserve θ. $\mathscr{D}^s_\theta(T^*M\backslash O)$ is not a closed subgroup in the group of H^s-diffeomorphisms of $T^*M\backslash O$ but in [17] it was shown that $\mathscr{D}_\theta^s(T^*M\backslash O)$ is a Hilbert manifold which makes $\mathscr{D}_\theta(T^*M\backslash O)$ into an ILH-Lie group. In Sect. 3 we review the basic facts on ILH-Lie groups and recall the construction of the ILH-Lie group structure for $\mathscr{D}_\theta(T^*M\backslash O)$.

The rest of this paper is devoted to giving the other end of the sequence (1.1), i.e. the invertible formal pseudodifferential operators, a Lie group structure and constructing a local section of the fibration $\pi:(\mathrm{FIO})_*\to\mathscr{D}_\theta(T^*M\backslash O)$. In the following paper [2] we describe how to patch these structures together to give a Lie group structure to the invertible formal Fourier integral operators.

To describe the Lie group structure of the invertible formal pseudodifferential operators it is convenient to restrict attention to the quotient $(\psi DO_{0,k})_*$ of invertible formal pseudodifferential operators of order zero modulo those of order $-k-1$. This gives a new exact sequence

$$I\to(\psi DO_{0,k})_*\xrightarrow{j}(\mathrm{FIO}_{0,k})_*\xrightarrow{\pi}\mathscr{D}_\theta(T^*M\backslash O)\to e\,, \tag{1.2}$$

where $(\mathrm{FIO}_{0,k})_*$ is the group of invertible Fourier integral operators of order zero modulo those of order $-k-1$. In Sect. 5 we will show that $(\psi DO_{0,k})_*$ is an ILH-Lie group and then define the differentiable structure on invertible, order zero, formal pseudodifferential operators by a direct limit procedure. In order to define the ILH-Lie group structure of $(\psi DO_{0,k})_*$ we need a global phase function for the identity map on $T^*M\backslash O$; this gives a global writing for pseudodifferential operators (modulo smoothing operators). In Sect. 4 we not only construct a global phase function for the identity map but in fact we construct a global phase function for all θ-preserving diffeomorphisms of $T^*M\backslash O$ which are C^1-close to the identity. This gives a global writing for invertible Fourier integral operators (modulo smoothing) with canonical relation close to the identity, thus providing a local section of the fibration $\pi:(\mathrm{FIO}_{0,k})_*\to\mathscr{D}_\theta(T^*M\backslash O)$.

2. Formal Pseudodifferential and Fourier Integral Operators

For a compact manifold M let $I^m(M,C)$ denote the space of Fourier integral operators of order m on M associated with the homogeneous canonical relation $C \subset(T^*M\backslash O)\times(T^*M\backslash O)$. The submanifold C in $(T^*M\backslash O)\times(T^*M\backslash O)$ is conic and Lagrangian with respect to the canonical symplectic form $\omega\ominus\omega=p_1^*\omega-p_2^*\omega$ (here p_j: $T^*M\times T^*M\to T^*M$, $j=1,2$, are projections onto the first and second factors and $\omega=-d\theta$ where θ is the canonical one form on T^*M). This means that $((x,\xi),(y,\eta))\in C$ implies $((x,\tau\xi),(y,\tau\eta))\in C$ for all $\tau>0$ and that $i^*(\omega\ominus\omega)\equiv 0$, where i: $C\to T^*M\times T^*M$ is inclusion. We consider $A\in I^m(M,C)$ as a continuous linear operator $A:C^\infty(M)\to C^\infty(M)$ which extends continuously to $\mathscr{E}'(M)\to\mathscr{E}'(M)$ and can be represented in local charts by the standard form

$$Au(x)=(2\pi)^{-n}\iint e^{i\varphi(x,y,\theta)}a(x,y,\theta)u(y)\,dy\,d\theta \tag{2.1}$$

with $x,y\in\Omega\subset\mathbb{R}^n$, $\theta\in\mathbb{R}^N\backslash O$, and $u\in C_c^\infty(\Omega)$.

In (2.1) $\varphi(x,y,\theta)$ is a nondegenerate phase function associated with C, and the amplitude $a(x,y,\theta)$ is an element of the space $S^{m-\frac{1}{2}(N-n)}(\Omega\times\Omega,\mathbb{R}^N)$, both in the sense of Duistermaat [5], Hörmander [10], Taylor [20] or Trèves [21]. This means explicitly

i) φ is a real valued smooth function on $\Omega \times \Omega \times \mathbb{R}^N \backslash O$ homogeneous of degree one in θ, i.e. $\varphi(x, y, \tau\theta) = \tau\varphi(x, y, \theta)$ for all $\tau > 0$.

ii) $d_{x,\theta}\varphi$ and $d_{y,\theta}\varphi$, the differentials of φ with respect to (x, θ) and (y, θ) respectively, do not vanish on the conic support of a.

iii) φ is nondegenerate, i.e. if $d_\theta\varphi(x, y, \theta) = 0$ and $(x, y, \theta) \in$ conic support of $a(x, y, \theta)$ then $d_{x,y,\theta}\left(\frac{\partial\varphi}{\partial\theta_j}(x, y, \theta)\right)$ are linearly independent for $j = 1, \ldots, N$. Nondegeneracy implies that $(x, y, \theta) \mapsto (x, y, d_{x,y}\varphi(x, y, \theta))$ is a diffeomorphism from the conic submanifold $C_\varphi = \{(x, y, \theta) \in$ conic support of $a \mid d_\theta\varphi(x, y, \theta) = 0\}$ onto an open cone in C.

iv) $a(x, y, \theta)$ is smooth on $\Omega \times \Omega \times \mathbb{R}^N$ satisfying the following property which defines $S^{m-\frac{1}{2}(N-n)}(\Omega \times \Omega, \mathbb{R}^N)$. For every compact subset K of $\Omega \times \Omega$ and every triplet of n-tuples α, β, γ there is a constant $C_{\alpha,\beta,\gamma}(K) > 0$ such that

$$|D_\theta^\alpha D_x^\beta D_y^\gamma a(x, y, \theta)| \leqq C_{\alpha,\beta,\gamma}(K)\,(1 + |\theta|)^{m - \frac{1}{2}(N-n) - |\alpha|} \tag{2.2}$$

for all $(x, y) \in K, \theta \in \mathbb{R}^N$.

For future reference we denote $I^{-\infty}(M, C) = \bigcap_{m \in \mathbb{R}} I^m(M, C)$ and $S^{-\infty}(\Omega \times \Omega, \mathbb{R}^N) = \bigcap_{m \in \mathbb{R}} S^m(\Omega \times \Omega, \mathbb{R}^N)$.

When two Fourier integral operators, $A_1 \in I^{m_1}(M, C_1)$ and $A_2 \in I^{m_2}(M, C_2)$ are composed the result is again a Fourier integral operator as long as the product $C_1 \times C_2$ intersects the diagonal $\{(x, y, y, z) \in (T^*M\backslash O) \times (T^*M\backslash O) \times (T^*M\backslash O) \times (T^*M\backslash O)\}$ transversally. In this case $C_1 \circ C_2 = \{(x, z) \in T^*M\backslash O \times T^*M\backslash O \mid \exists y \in T^*M\backslash O$ with $(x, y) \in C_1$ and $(y, z) \in C_2\}$ is a smooth canonical relation and $A_1 \circ A_2 \in I^{m_1+m_2}(M, C_1 \circ C_2)$. More generally one can allow clean intersections of canonical relations but since we are concerned only with invertible Fourier integral operators it is no restriction to consider only tranverse intersections. Indeed, if the inverse of $A \in I^m(M, C)$ is again to be a Fourier integral operator then the canonical relation C must be invertible, that is, C must be the graph of a diffeomorphism $\eta : T^*M\backslash O \to T^*M\backslash O$. That C is canonical implies that η must be a symplectomorphism, i.e. $\eta^*\omega = \omega$, and that C is conic implies that η is homogeneous of degree one, i.e. $\eta(\tau\alpha_x) = \tau\eta(\alpha_x)$, $\tau > 0$. Indeed, if C is conic then $(\alpha_x, \beta_y) \in C$ implies $(\tau\alpha_x, \tau\beta_y) \in C$ so if $\eta(\alpha_x) = \beta_y$ then $\eta(\tau\alpha_x) = \tau\beta_y = \tau\eta(\alpha_x)$. It is well known (e.g. Hörmander [10], Weinstein [22]) that a diffeomorphism of $\eta : T^*M\backslash O \to T^*M\backslash O$ is symplectic and homogeneous of degree one if and only if η preserves the canonical one form, i.e. $\eta^*\theta = \theta$. Note that for this to be true it is crucial that the zero section is deleted from T^*M, otherwise η would just be a lift of a diffeomorphism $g : M \to M$.

Let $\mathcal{D}_\theta(T^*M\backslash O) = \{\eta : T^*M\backslash O \to T^*M\backslash O \mid \eta$ is a diffeomorphism and $\eta^*\theta = \theta\}$ be the group of homogeneous canonical diffeomorphisms of $T^*M\backslash O$. Denote the class of Fourier integral operators of order m with $C = \mathrm{graph}(\eta)$ by $I^m(M, \eta)$. If $A_1 \in I^{m_1}(M, \eta_1)$ and $A_2 \in I^{m_2}(M, \eta_2)$ then $A_1 \circ A_2 \in I^{m_1+m_2}(M, \eta_1 \circ \eta_2)$ since $\mathrm{graph}(\eta_1) \circ \mathrm{graph}(\eta_2) = \mathrm{graph}(\eta_1 \circ \eta_2)$.

The class of pseudodifferential operators of order m on M is defined as $L^m(M) = I^m(M, e)$ where e is the identity $e : T^*M\backslash O \to T^*M\backslash O$. A pseudodifferential operator P can be locally represented in the form

$$Pu(x)=(2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)\,dy\,d\xi\,, \tag{2.3}$$

where $x, y\in\Omega\subset\mathbb{R}^n$, $\xi\in\mathbb{R}^n$ and $a(x,y,\xi)\in S^m(\Omega\times\Omega,\mathbb{R}^n)$.

The Schwartz kernel theorem states that for any continuous map $K: C^\infty(M)\to\mathcal{D}'(M)$, there exists a unique distribution $\hat{K}\in\mathcal{D}'(M\times M)$ called the *Schwartz Kernel* such that for all $\psi,\varphi\in C^\infty(M)$, $\langle K\varphi,\psi\rangle=\langle\hat{K},\varphi\otimes\psi\rangle$. Here and in what follows $C^\infty(M)$ and $\mathcal{D}'(M)$ are equipped with their standard seminorms with respect to which they are Fréchet spaces. A continuous linear operator from $C^\infty(M)$ to $\mathcal{D}'(M)$ is called *regularizing* (or *smoothing*), if it extends to a continuous linear map of $\mathcal{D}'(M)$ into $C^\infty(M)$. This happens if and only if the associated Schwartz Kernel is C^∞ on $M\times M$. The spaces $L^{-\infty}(M)=\bigcap_{m\in\mathbb{R}} L^m(M)$ and $I^{-\infty}(M,\eta)=\bigcap_{m\in\mathbb{R}} I^m(M,\eta)$ both coincide with the space of regularizing operators on M (see e.g. Trèves [21]).

In this paper we shall consider only *formal* pseudodifferential and Fourier integral operators: These are defined as follows. Let $S^m(\Omega)$ be the subspace of $S^m(\Omega\times\Omega,\mathbb{R}^n)$ consisting of amplitudes $a(x,\xi)$ independent of y. The map associating to $p(x,\xi)\in S^m(\Omega)$ the operator $P\in L^m(\Omega)$ given by

$$Pu(x)=(2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}p(x,\xi)u(y)\,dy\,d\xi \tag{2.4}$$

induces an isomorphism of $S^m(\Omega)/S^{-\infty}(\Omega)$ with $L^m(\Omega)/L^{-\infty}(\Omega)$ [26]. Elements of $S^m(\Omega)$ are called *symbols* of order m. A symbol is called *classical* if it has an asymptotic expansion $\sum_{j=0}^{+\infty} a_{m-j}(x,\xi)$ where each term $a_{m-j}(x,\xi)$ is smooth off $\xi=0$ and is positive homogeneous of degree $m-j$ in ξ, i.e. $a_{m-j}(x,\tau\xi)=\tau^{m-j}a_{m-j}(x,\xi)$ for all $\tau>0$. (To say that $a(x,\xi)$ has the asymptotic expansion $\sum_{j=0}^{\infty} a_{m-j}(x,\xi)$ means that

$$(1-\chi(x,\xi))\left(a(x,\xi)-\sum_{j=0}^{k-1} a_{m-j}(x,\xi)\right)\in S^{m-k}(\Omega)$$

for all k where $\chi(x,\xi)$ is a bump function on $\Omega\times\mathbb{R}^n$ which has the value one on a neighborhood of $\Omega\times\{0\}$ and is compactly supported in the ξ direction.) Since the asymptotic expansion is independent of changes in $a(x,\xi)$ by elements of $S^{-\infty}(\Omega)$ it makes sense to talk about *classical pseudodifferential operators* on Ω as pseudodifferential operators on Ω which can be written in the form (2.4) modulo a smoothing operator where $p(x,\xi)$ is a classical symbol. For formal manipulations it is convenient to ignore the smoothing part of P. In particular the equivalence classes of classical pseudodifferential operators in $L^m(\Omega)/L^{-\infty}(\Omega)$ are in one to one correspondence with the space of asymptotic expansions $\sum_{j=0}^{\infty} a_{m-j}(x,\xi)$. This leads to the definition: A *formal pseudodifferential operator* on M of order m is an element of $L^m(M)/L^{-\infty}(M)$ which has a representative in any local chart Ω of the form (2.4) with $p(x,\xi)$ defining a classical symbol. We denote the space of formal pseudodifferential operators of order m by ψDO_m. The *principal symbol* $\sigma(P)(x,\xi)$

of $P \in \psi DO_m$ equals $a_m(x, \xi)$ in any local chart and is globally defined as a smooth homogeneous function of degree m on $T^*M \backslash O$.

A *formal Fourier integral operator* of order m, with canonical relation graph(η), is an element of the quotient space $I^m(M, \eta)/I^{-\infty}(M, \eta)$ which has a representative in any local chart Ω of the form (2.1) with $a(x, y, \xi)$ defining a classical symbol. We call this space $\mathrm{FIO}_m(\eta)$ and denote

$$\mathrm{FIO}_m = \bigcup_{\eta \in \mathscr{D}_\theta(T^*M \backslash O)} \mathrm{FIO}_m(\eta).$$

Notice that for $-k \leqq m$ we have $\psi DO_{-k-1} \subset \psi DO_m$. We let

$$\psi DO_{m,k} = \psi DO_m / \psi DO_{-k-1},$$

these are the formal pseudodifferential operators of order m, modulo those of order $-k-1$. Note that on Ω there is an isomorphism between $\psi DO_{m,k}$ and the classical symbols in $S^m(\Omega)/S^{-k-1}(\Omega)$, which in turn can be thought of as finite expansions $\sum_{j=1}^{m-k} a_{m-j}(x, \xi)$ where $a_{m-j}(x, \xi)$ is homogeneous of degree $m-j$. Likewise we define $\mathrm{FIO}_{m,k}(\eta) = \mathrm{FIO}_m(\eta)/\mathrm{FIO}_{-k-1}(\eta)$ the formal Fourier integral operators of order m with canonical relation graph(η) modulo those of order $-k-1$. In any local chart Ω we can think of an element of $\mathrm{FIO}_{m,k}(\eta)$ as the equivalence class of an operator of the form (2.1) where $a(x, y, \theta)$ has a finite expansion $\sum_{j=1}^{m+k} a_{m_0-j}(x, y, \theta)$ where $m_0 = m - \frac{1}{2}(N-n)$. Of course $\mathrm{FIO}_{m,k}(e) = \psi DO_{m,k}$, where e is the identity. Finally, we let

$$\mathrm{FIO}_{m,k} = \bigcup_{\eta \in \mathscr{D}_\theta(T^*M \backslash O)} \mathrm{FIO}_{m,k}(\eta).$$

Notice that if $m=0$ then composition is well defined. Indeed, let $[A]$, $[B] \in \mathrm{FIO}_{0,k}$ and let A and B be representatives of these classes, then $[A \circ B]$ is independent of the choice of representatives since if $R_1, R_2 \in \mathrm{FIO}_{-k-1}$ then $(A+R_1) \circ (B+R_2) = A \circ B + R_1 \circ B + A \circ R_2 + R_1 \circ R_2$ and $R_1 \circ B + A \circ R_2 + R_1 \circ R_2 \in \mathrm{FIO}_{-k-1}$. We write $[A] \circ [B] = [A \circ B]$. Of course if $[A], [B] \in \psi DO_{0,k}$ then $[A] \circ [B] \in \psi DO_{0,k}$ as well since in general if $[A] \in \mathrm{FIO}_{0,k}(\eta_1)$ and $[B] \in \mathrm{FIO}_{0,k}(\eta_2)$ then $[A] \circ [B] \in \mathrm{FIO}_{0,k}(\eta_1 \circ \eta_2)$. For $[A]$, $[B] \in \psi DO_{0,k}$ the asymptotic formula for the symbol of $A \circ B$ gives that the symbol for $[A] \circ [B]$ is

$$\sum_{j=0}^{k} c_{-j}(x, \xi),$$

where

$$c_{-j}(x, \xi) = \sum_{r+s-|\alpha| = -j} D_x^\alpha a_r(x, \xi) \partial_\xi^\alpha b_s(x, \xi).$$

Let $(\mathrm{FIO}_{0,k})_*$ and $(\psi DO_{0,k})_*$ denote the groups of invertible elements of $\mathrm{FIO}_{0,k}$ and $\psi DO_{0,k}$ respectively.

Proposition 2.1. *The following sequence of groups is exact*

$$I \to (\psi DO_{0,k})_* \xhookrightarrow{j} (\mathrm{FIO}_{0,k})_* \xrightarrow{\pi} \mathscr{D}_\theta(T^*M \backslash O) \to e.$$

Proof. The map j is the inclusion. The map π which associates to any $[A]\in(\mathrm{FIO}_{0,k})_*$ its canonical relation is by previous remarks a surjective group homomorphism i.e. $\pi([A])=\eta\in\mathscr{D}_\theta(T^*M\backslash O)$ if and only if $[A]\in(\mathrm{FIO}_{0,k}(\eta))_*$. The kernel of π consists of elements in $(\mathrm{FIO}_{0,k})_*$ with canonical relation given by the identity, i.e. $\ker(\pi)=(\psi DO_{0,k})_*$. □

3. ILB-Lie Groups and $\mathscr{D}_\theta(T^*M\backslash O)$

In this section we describe what is meant by the "Lie group" structure of diffeomorphism groups. This will also be the setting in which the groups of invertible, zero order, pseudodifferential operators and Fourier integral operators are Lie groups.

Definition 3.1. A collection of topological vector spaces $\{E^\infty, E^s | s\geqq s_0\}$ is called a *nested Banach space* if

(i) each $E^s, s\geqq s_0, s\neq\infty$ is a Banach space;

(ii) for each $s\geqq s_0$ there are linear continuous dense inclusions $E^{s+1}\hookrightarrow E^s$;

(iii) $E^\infty=\bigcap_{s\geqq s_0} E^s$ with the inverse limit topology.

Note that E^∞ is a graded Fréchet space [9, p. 133]. Since every Fréchet space is an inverse limit of Banach spaces, one can always consider the nested Banach space associated to a given Fréchet space. Nested Banach spaces serve as models for nested manifolds.

Definition 3.2. A collection of topological spaces $\{M^\infty, M^s | s\geqq s_0\}$ is called a *nested Banach manifold* modeled on the nested Banach space $\{E^\infty, E^s | s\geqq s_0\}$ if

(i) each $M^s, s\geqq s_0, s\neq\infty$, is a Banach manifold of class $C^{k(s)}$ modeled on E^s, where the order of differentiability $k(s)$ is strictly increasing as $s\to\infty$;

(ii) for each $s\geqq s_0, s\neq\infty$, there exist dense inclusions $M^{s+1}\hookrightarrow M^s$ of class $C^{k(s)}$;

(iii) $M^\infty=\bigcap_{s\geqq s_0} M^s$ with the inverse limit topology;

(iv) if (U^s,φ^s) is a chart on $M^s, s\geqq s_0, s\neq\infty$, then $(U^s\cap M^t, \varphi^s|_{U^s\cap M^t})$ is a chart on M^t for all $t\geqq s$.

Note that condition (iv) is stronger than (ii). It says that in spite of the fact that M^t is *not* a submanifold of M^s for $t\geqq s$, charts on M^s do induce charts on M^t. In applications, this hypothesis becomes essential in proofs of regularity theorems. An important consequence of the definition is that M^∞ is a Fréchet manifold, modeled on E^∞; the charts are $(U^s\cap M^\infty, \varphi^s|_{U^s\cap M^\infty})$ for any $s\geqq s_0$. (The definition we adopt for C^k maps between Fréchet spaces is the following: $D^k f(x)(h_1,\dots,h_k)$ is jointly continuous in $x, h_1,\dots,h_k$; see [9] for details.)

Nested Banach spaces and manifolds are also called ILB (inverse limit of Banach) spaces and manifolds; this terminology is due to Omori [15] and emphasizes E^∞ and M^∞ respectively. In our point of view, however, the study of M^s is emphasized, the properties of M^∞ being corollaries. To prove the theorems about M^∞, one can take different points of view. One point of view is to attempt to systematically use the powerful Nash-Moser implicit function theorem in the tame category [9]. For many purposes, however, a second point of view is often useful. One applies the ordinary implicit function theorem on M^s and proves in addition a regularity theorem that allows the passage $s \to \infty$, without shrinking the interval of time existence, or the neighborhood in question, to a point. This approach enables one to obtain results just as good as with the Nash-Moser theorem; [7, 13]. Of course the applications of the two points overlap but are by no means equal.

Definition 3.3. Let $\{M^\infty, M^s | s \geqq s_0\}$ and $\{N^\infty, N^t | t \geqq t_0\}$ be two nested manifolds. A map $f: M^\infty \to N^\infty$ is a C^k *nested map*, $k \geqq 0$, if there is a $t_f \geqq t_0$ such that for every $t \geqq t_f$ there exists $s(t) \geqq s_0$ and a C^k-extension $f^t: M^{s(t)} \to N^t$ of f. A map $f: M^\infty \to N^\infty$ is *smooth*, or C^∞ *nested*, if it is C^k nested for every $k \geqq 0$. (Here, t_f is allowed to depend on k.) A map $f: M^\infty \to N^\infty$ is a C^k *nested diffeomorphism*, $0 \leqq k \leqq \infty$, if it has a C^k nested inverse $g: N^\infty \to M^\infty$.

In this definition note that $k \leqq k(t_f)$, $k \leqq k(s(t_f))$, since the order of smoothness of the manifold must exceed or be equal to k. Also note that in spite of the fact that M^s and N^t are in general not C^∞ manifolds, we can define C^∞ maps between the nested manifold $\{M^\infty, M^s | s \geqq s_0\}$, $\{N^\infty, N^t | t \geqq t_0\}$.

Definitions 3.2 and 3.3 define the category of nested manifolds. We shall be concerned in this paper exclusively with infinite dimensional nested manifolds. Let us recall the standard example. Let M and N be smooth finite dimensional manifolds with M compact, possibly with boundary and N boundaryless. Let $C^\infty(M, N)$, $H^s(M, N)$ denote the C^∞ and Sobolev class H^s maps from M into N, $s > (\dim M)/2$. $H^s(M, N)$ is a C^∞ Hilbert manifold whose tangent space at f equals

$$T_f(H^s(M, N)) = \{X \in H^s(M, TN) | \tau_N \circ X = f\},$$

where $\tau_N: TN \to N$ is the canonical tangent bundle projection. Thus $T_f(H^s(M, N))$ consists of H^s-vector fields over $f \in H^s(M, N)$. A chart at f in $H^s(M, N)$ is given by the exponential map of a Riemannian metric on N [6, 16, 14]. In this way $\{C^\infty(M, N), H^s(M, N) | s > (\dim M)/2\}$ is a nested Hilbert manifold.

An example of a smooth nested map often encountered is the following. Let $H^s(\Lambda^k(M))$ denote the Hilbertizable space of all H^s k-forms on the compact manifold M and let $C^\infty(\Lambda^k(M)) = \Omega^k(M)$ be the smooth sections of the bundle of k-forms $\Lambda^k(M)$. Then for $\alpha \in \Omega^k(N)$ define the map $\psi_\alpha: C^\infty(M, N) \to \Omega^k(M)$ by $\psi_\alpha(f) = f^*\alpha$. This map is a C^∞ nested map between $\{C^\infty(M, N), H^s(M, N) | s > \dim M/2 + 1\}$ and $\{\Omega^k(M), H^s(\Lambda^k(M)) | s > (\dim M)/2\}$ since $\psi_\alpha: H^{s+1}(M, N) \to H^s(\Lambda^k(M))$ is C^∞; see [6].

In these examples the manifolds M^s, N^t were C^∞ Banach manifolds. This is not the case for the manifold structures of the groups of invertible pseudodifferential and Fourier integral operators, as we shall see in the course of this paper and the forthcoming paper [2]. The definition of the topology will impose restrictions on the order of differentiability.

With the concept of nested manifolds and nested maps one can define tangent bundles, bundles of k-forms, nested Lie groups and nested Lie algebras, just by following the standard definitions. But as is well-known, diffeomorphism groups have a much finer structure than that of a nested Lie group. The following definition weakens Omori's concept of an ILB-Lie group which is adapted to the only known examples, the diffeomorphism groups. In addition, as we shall see, this definition encompasses also the Lie group structures of invertible pseudodifferential and Fourier integral operators.

Definition 3.4. An ILB-*Lie group* $\{G^\infty, G^s | s \geqq s_0\}$ is a nested manifold with G^∞, G^s topological groups (in the manifold topology), satisfying the following additional conditions:

(i) the group multiplication $G^\infty \times G^\infty \to G^\infty$ can be extended to a C^k-map $G^{s+k} \times G^s \to G^s$ for every s satisfying $k \leqq k(s)$;

(ii) the inversion mapping $G^\infty \to G^\infty$ can be extended to a C^k-map $G^{s+k} \to G^s$, for every s satisfying $k \leqq k(s)$;

(iii) right translation R_g by $g \in G^s$, $R_g(h) = hg$, is a $C^{k(s)}$-map, $R_g : G^s \to G^s$.

If G^s is the group $\mathcal{D}^s(M)$ of H^s diffeomorphisms of a compact manifold, then $k(s) = \infty$ and the nested Lie group $\{\mathcal{D}^\infty(M), \mathcal{D}^s(M) | s \geqq (\dim M)/2 + 1\}$ satisfies the following additional property:

The tangent map of right translation

$$TR : \mathfrak{g}^{s+k} \times G^s \to TG^s, TR(g, \xi) = T_e R_g(\xi),$$

where

$$\mathfrak{g}^{s+k} = T_e G^{s+k}, \text{ is a } C^k \text{ map}.$$

We shall point out below what consequences such a property has in the definition of Lie algebras. In the general case, when G^s is only of class $k(s)$, the tangent bundle is of class $C^{k(s)-1}$ and one expects the loss of a derivative. Indeed, from property (i) it follows easily that TR is of class C^{k-1}.

Note that condition (iv) in Definition 3.2 is equivalent in the case of ILB-Lie groups to the following: *there are open neighborhoods U of e in G^{s_0} and V of 0 in $\mathfrak{g}^{s_0}$ and a diffeomorphism $\varphi: U \to V$ such that $\varphi|_{U \cap G^s}$ is a diffeomorphism of $U \cap G^s$ in the manifold structure of G^s with the open subset $V \cap \mathfrak{g}^s$ in $\mathfrak{g}^s$ for all $s \geqq s_0$.*

Proposition 3.5. *Let $\{G^\infty, G^s | s \geqq s_0\}$ be an* ILB-*Lie group and let $\mathfrak{g}^s = T_e G^s$. For $\xi \in \mathfrak{g}^s$, let X_ξ denote the right invariant vector field $X_\xi(g) = T_e(R_g(\xi))$ on G^s. The operation*

$$[\xi, \eta] = -[X_\xi, X_\eta](e)$$

for $\xi \in \mathfrak{g}^{s+2}$, $\eta \in \mathfrak{g}^{t+2}$ is a bilinear, continuous, antisymmetric map $\mathfrak{g}^{s+2} \times \mathfrak{g}^{t+2} \to \mathfrak{g}^{\min(s,t)}$, it satisfies the Jacobi identity in $\mathfrak{g}^{\min(s,t,r)}$ for elements in $\mathfrak{g}^{s+4}$, $\mathfrak{g}^{t+4}$, $\mathfrak{g}^{r+4}$.

Proof. Since $TR : \mathfrak{g}^{s+2} \times G^s \to TG^s$ is C^1, X_ξ is a C^1 vector field on G^s. Thus, on $G^{\min(s,t)}$, $[X_\xi, X_\eta]$ is a well-defined vector field on the densely embedded manifold $G^{\max(s,t)}$; this vector field is of class C^0. All other properties are easily verified. The minus sign in the definition of the bracket appears since the Lie algebra bracket is usually thought of as being defined by left invariant vector fields, an operation that does not have enough smoothness in the present context. □

Motivated by this proposition, we state the following

Definition 3.6. An ILB-*Lie algebra* is a nested Banach space $\{\mathfrak{g}^\infty, \mathfrak{g}^s | s \geqq s_0\}$ with bilinear, continuous antisymmetric maps $\mathfrak{g}^{s+2} \times \mathfrak{g}^{t+2} \to \mathfrak{g}^{\min(s,t)}$ for all $s, t \geqq s_0$ which satisfy the Jacobi identity on $\mathfrak{g}^{\min(s,t,r)}$ for elements in $\mathfrak{g}^{s+4} \times \mathfrak{g}^{t+4} \times \mathfrak{g}^{r+4}$.

The classical examples of ILB-Lie groups are the diffeomorphism groups. The ILB-Lie algebra of the ILB-Lie groups $\{\mathcal{D}^\infty(M), \mathcal{D}^s(M) | s > (\dim M)/2 + 1\}$ is $\{\mathcal{X}(M), H^s(TM) | s > (\dim M)/2 + 1\}$, where $\mathcal{X}(M)$ denotes the C^∞ vector fields on

M and $H^s(TM)$ the H^s vector fields on M. In this case, as already mentioned, each $\mathscr{D}^s(M)$ is a C^∞ Hilbert manifold. In Proposition 3.5, all orders of differentiability drop by one, i.e. we get maps $\mathfrak{g}^{s+1} \times \mathfrak{g}^{t+1} \to \mathfrak{g}^{\min(s,t)}$, etc; this is due to the fact that $TR: \mathfrak{g}^{s+k} \times G^s \to TG^s$ is in this particular case C^k and not merely C^{k-1}. The Lie algebra bracket is *minus* the usual Lie bracket operation of vector fields [7, 15]. If the models of all G^s in Definition 3.4 are Hilbert spaces, $\{G^\infty, G^s | s \geqq s_0\}$ is called an ILH-(inverse limit of Hilbert) *Lie group*. In particular, all diffeomorphism groups are ILH-Lie groups. In the course of this paper we shall often refer to G^∞ or G^s as Lie groups and understand that we mean $\{G^\infty, G^s | s \geqq s_0\}$ as ILB or ILH-Lie groups.

3.7. *The Lie group* $\mathscr{D}^s_\theta(T^*M \backslash O)$. The group $\mathscr{D}_\theta(T^*M \backslash O)$ of diffeomorphisms of the cotangent bundle T^*M minus the zero section, preserving the canonical one-form θ of T^*M, will play a crucial role in the definition of a Lie group structure for $(\mathrm{FIO})_*$. Referring to the prior discussion of manifolds of maps, it should be noted that this group has an added difficulty in that $T^*M \backslash O$ is *not* compact, so the standard results on diffeomorphism groups do not apply in a straightforward manner. We shall review here, following [17], how $\mathscr{D}^{s+1}_\theta(T^*M \backslash O)$ is made into a Lie group with Lie algebra $\mathscr{S}^{s+2}_\theta(T^*M \backslash O) = \{H \in H^{s+2}(T^*M \backslash O, \mathbb{R}) | H$ homogeneous of degree one$\}$ with the Poisson bracket as Lie algebra bracket, $s > \dim M + 1/2$. The main idea is that $\mathscr{D}^{s+1}_\theta(T^*M \backslash O)$ is algebraically isomorphic to the Lie group of all H^{s+1} contact transformations of the cosphere bundle of M, which is a compact manifold. We start by recalling the relevant facts.

The multiplicative group of strictly positive reals $\mathbb{R}_+$ acts smoothly on $T^*M \backslash O$ by $\alpha_x \to \tau\alpha_x$, $\tau > 0$, $\alpha_x \in T^*_xM$, $\alpha_x \neq 0$. This action is free and proper and therefore $\pi: T^*M \backslash O \to Q \equiv (T^*M \backslash O)/\mathbb{R}_+$ is a smooth principal fiber bundle over Q, the *cosphere bundle* of M. Note that Q is compact and odd-dimensional. Also Q carries a canonical contact structure given by projecting the sub-bundle $\operatorname{Ker}\theta \subset T(T^*M)$ down to a sub-bundle $E \subset TQ$ by the differential of the projection $\pi: T^*M \to Q$. A *contact one-form* on Q is a one-form α on Q with $\operatorname{Ker}\alpha = E$. Q carries no canonical contact one-form but for each global section $\sigma: Q \to T^*M \backslash O$ we can define an exact contact one-form θ_σ on Q by $\theta_\sigma = \sigma^*\theta$. Such global sections exist in abundance; for example, any Riemannian metric on M identifies T^*M with TM and Q with the unit sphere bundle. Then the usual inclusion of the sphere bundle into TM gives a section σ. The section σ is uniquely determined by a smooth function $f_\sigma: T^*M \backslash O \to \mathbb{R}_+$ defined by $\sigma(\pi(\alpha_x)) = f_\sigma(\alpha_x)\alpha_x$. In other words, f_σ measures how far from the section σ an element $\alpha_x \in T^*M \backslash O$ lies. The function f_σ is homogeneous of degree -1 and $\pi^*\theta_\sigma = f_\sigma\theta$.

An H^{s+1} *contact transformation* on Q is a diffeomorphism $\varphi \in \mathscr{D}^{s+1}(Q)$ such that for any two sections $\sigma, \zeta: Q \to T^*M \backslash O$, there exists an H^{s+1} function $h_{\sigma\zeta}: Q \to \mathbb{R}_+$ satisfying $\varphi^*\theta_\sigma = h_{\sigma\zeta}\theta_\zeta$. Equivalently, $\varphi \in \mathscr{D}^{s+1}(Q)$ is a H^{s+1} contact transformation if and only if for each global section σ there exists an H^{s+1} function $h_\sigma: Q \to \mathbb{R}_+$ such that $\varphi^*\theta_\sigma = h_\sigma\theta_\sigma$. The function h_σ is uniquely determined by σ, namely $h_\sigma = \langle \varphi^*\theta_\sigma, E_\sigma \rangle$, where E_σ is the Reeb vector field on Q determined by the contact structure θ_σ and $\langle\ ,\ \rangle$ denotes the pairing between vector fields and one forms. (E_σ is the unique vector field satisfying $\langle \theta_\sigma, E_\sigma \rangle = 1$ and $i_{E_\sigma}(d\theta_\sigma) = 0$, where $i_{E_\sigma}(d\theta_\sigma)$ denotes the interior product of E_σ with $d\theta_\sigma$; in local coordinates $(x^1, \ldots, x^{n-1}, y^1, \ldots, y^{n-1}, t)$ on Q, where $\theta_\sigma = \sum_{i=1}^{n-1} y^i dx^i + dt$ we have $E_\sigma = \left(\dfrac{\partial}{\partial t}\right)$.

Therefore the group of H^{s+1}-contact transformations on Q is isomorphic to the group

$$\mathrm{Con}_\sigma^{s+1}(Q)=\{(\varphi,h)\in\mathscr{D}^{s+1}(Q)\ltimes H^{s+1}(Q,\mathbb{R}\backslash O)|\varphi^*\theta_\sigma=h\theta_\sigma\}$$

for any fixed but arbitrary global section σ, where $\mathscr{D}^{s+1}(Q)\ltimes H^{s+1}(Q,\mathbb{R}\backslash O)$ is the semidirect product of the Lie groups $\mathscr{D}^{s+1}(Q)$ and $H^{s+1}(Q,\mathbb{R}\backslash O)$ ($H^{s+1}(Q,\mathbb{R}\backslash O)$ as a multiplicative group) with composition law $(\varphi_1,h_1)\cdot(\varphi_2,h_2)=((\varphi_1\circ\varphi_2), h_2(h_1\circ\varphi_2))$. Omori [15] has shown that $\mathrm{Con}_\sigma^{s+1}(Q)$ is a closed Lie subgroup of the Lie group $\mathscr{D}^{s+1}(Q)\ltimes H^{s+1}(Q,\mathbb{R}\backslash O)$. The Lie algebra of $\mathscr{D}^{s+1}(Q)\ltimes H^{s+1}(Q,\mathbb{R}\backslash O)$ is the semidirect product $\mathscr{X}^{s+1}(Q)\ltimes H^{s+1}(Q,\mathbb{R})$ of H^{s+1}-vector fields and H^{s+1} functions, with bracket $[(X,f),(Y,g)]=([X,Y],X(g)-Y(f))$. The Lie algebra of $\mathrm{Con}_\sigma^{s+1}(Q)$ is $\mathrm{con}_\sigma^{s+1}(Q)=\{(Y,g)\in\mathscr{X}^{s+1}(Q)\ltimes H^{s+1}(Q,\mathbb{R})|L_Y\theta_\sigma=g\theta_\sigma\}$ (L_Y denoting the Lie derivative along the vector field Y).

In [17, Theorem 4.1] it is shown that the group $\mathscr{D}_\theta^{s+1}(T^*M\backslash O)$ is isomorphic (as a group) to the Lie group $\mathrm{Con}_\sigma^{s+1}(Q)$. The isomorphism is given by Φ: $\mathscr{D}_\theta^{s+1}(T^*M\backslash O)\to\mathrm{Con}^{s+1}(Q)$, $\Phi(\eta)=(\varphi,h)$ where φ is defined by $\varphi\circ\pi=\pi\circ\eta$ and h by $h\circ\pi=(f_\sigma\circ\eta)/f_\sigma$, $\sigma(\pi(\alpha_x))=f_\sigma(\alpha_x)\alpha_x$. The inverse of Φ is given by $\Phi^{-1}(\varphi,h)=(\sigma\circ\varphi\circ\pi)/(h\circ\pi)\cdot f_\sigma$.

Since $\mathrm{Con}_\sigma^{s+1}(Q)$ and $\mathrm{Con}_\zeta^{s+1}(Q)$ are isomorphic as ILH-Lie groups for any two global sections σ and ζ, the isomorphism Φ determines an ILH-Lie group structure on $\mathscr{D}_\theta^{s+1}(T^*M\backslash O)$ which is independent of σ, (or independent of the Riemannian metric if σ is induced from such). Furthermore, the Lie algebra of $\mathscr{D}_\theta^{s+1}(T^*M\backslash O)$, $\mathscr{X}_\theta^{s+1}(T^*M\backslash O)=\{Y\in\mathscr{X}^{s+1}(T^*M\backslash O)|L_Y\theta=0\}$ is isomorphic to $\mathscr{S}^{s+2}(T^*M\backslash O)=\{H\in H^{s+2}(T^*M\backslash O,\mathbb{R})|h$ homogeneous of degree one$\}$. This is because $L_Y\theta=0$ if and only if Y is globally Hamiltonian, homogeneous of degree zero, with Hamiltonian function $H=\theta(Y)$, homogeneous of degree one. Moreover, the Lie algebras $\mathscr{X}_\theta^{s+1}(T^*M\backslash O)$ and $\mathrm{con}_\sigma^{s+1}(Q)$ are isomorphic via $T_e\Phi:\mathscr{X}_\theta^{s+1}(T^*M\backslash O)\to\mathrm{con}_\sigma^{s+1}(Q)$. Explicitly, $T_e\Phi(X_H)=(X,k)$, where X is uniquely defined by $T\pi\circ X_H=X\circ\pi$ and k by $k\circ\pi=\{f_\sigma,H\}/f_\sigma$, ($\{\ ,\ \}$ is the canonical Poisson bracket on T^*M). The map $H\mapsto H\circ\sigma$ is an isomorphism from $\mathscr{S}^{s+2}(T^*M\backslash O)$ onto $H^{s+2}(Q,\mathbb{R})$, with inverse j: $H^{s+2}(Q,\mathbb{R})\to\mathscr{S}^{s+2}(T^*M\backslash O)$, where $j(f)$ is the extension to $T^*M\backslash O$ by homogeneity of degree one of $f\circ\sigma^{-1}:\sigma(Q)\subset(T^*M\backslash O)\to\mathbb{R}$, for $f\in H^{s+2}(Q,\mathbb{R})$. The composition of these two isomorphisms with $T_e\Phi^{-1}$ gives an isomorphism

$$F:\mathrm{con}_\sigma^{s+1}(Q)\xrightarrow{T_e\Phi^{-1}}\mathscr{X}_\theta^{s+1}(T^*M\backslash O)\to\mathscr{S}^{s+2}(T^*M\backslash O)\xrightarrow{j^{-1}}H^{s+2}(Q,\mathbb{R}),$$

$F(X,k)=\theta_\sigma(X)$. In the condition $L_X\theta_\sigma=k\theta_\sigma$, the function k is uniquely determined by X, namely, $k=E_\sigma(\theta_\sigma(X))$. From this it follows easily that F is continuous and hence an isomorphism between $\mathrm{con}_\sigma^{s+1}(Q)$ and $H^{s+2}(Q,\mathbb{R})$ (note the gain of one derivative). We see thus once again that $\mathrm{Con}_\sigma^{s+1}(Q)$ and $\mathrm{Con}_\zeta^{s+1}(Q)$ are isomorphic as ILH-Lie groups for any two global sections $\sigma,\zeta:Q\to T^*M\backslash O$, since both are modelled on $H^{s+2}(Q,\mathbb{R})$.

Defining the Hilbert space structure of $\mathscr{S}^{s+2}(T^*M\backslash O)$ as the one induced by the isomorphism j: $H^{s+2}(Q,\mathbb{R})\to\mathscr{S}^{s+2}(T^*M\backslash O)$, it follows that $H^{s+2}(Q,\mathbb{R})$, $\mathscr{S}^{s+2}(T^*M\backslash O)$, $\mathscr{X}_\theta^{s+1}(T^*M\backslash O)$ and $\mathrm{con}_\sigma^{s+1}(Q)$ are all isomorphic as Hilbert spaces. It is desirable to compare the topology of $\mathscr{S}^{s+2}(T^*M\backslash O)$ with the strong C^1-Whitney topology. Since all elements of $\mathscr{S}^{s+2}(T^*M\backslash O)$ are C^2 by the Sobolev imbedding theorem, we can define a new topology on $\mathscr{S}^{s+2}(T^*M\backslash O)$ in the following way: a neighborhood of zero consists of all those functions $H\in\mathscr{S}^{s+2}(T^*M\backslash O)$ for which $dH:(T^*M\backslash O)\to T^*(T^*M\backslash O)$ is C^1-close to zero in the strong C^1-Whitney topology. Denote by $\mathscr{S}_W^{s+2}(T^*M\backslash O)$ the space $\mathscr{S}^{s+2}(T^*M\backslash O)$ equipped with this new topology. It is then a routine matter to check that j: $H^{s+2}(Q,\mathbb{R})\to\mathscr{S}_W^{s+2}(T^*M\backslash O)$, or equivalently the identity $\mathscr{S}^{s+2}(T^*M\backslash O)\to\mathscr{S}_W^{s+2}(T^*M\backslash O)$, are continuous with discontinuous inverses, i.e. the new topology is strictly coarser than the original one on $\mathscr{S}^{s+2}(T^*M\backslash O)$. This remark will be useful in the construction of an explicit chart at e in $\mathscr{D}_\theta^{s+1}(T^*M\backslash O)$, which we will give in Sect. 4.

Remarks. (1) The gain of one derivative at the Lie algebra level has a corresponding statement in $\mathscr{D}_\theta^{s+1}(T^*M\backslash O)$: For every $\eta\in\mathscr{D}_\theta^{s+1}(T^*M\backslash O)$, $\tau^*\circ\eta:(T^*M\backslash O)\to M$ is of class H^{s+2}, where $\tau^*:T^*M\to M$ is the cotangent bundle projection. Locally, this means that if $\eta(x,\alpha)=(y(x,\alpha),\,\beta(x,\alpha))$, then y is H^{s+2} jointly in x and α. To prove this, note that $\eta^*\theta=\theta$ is equivalent locally to

$$\sum_{i=1}^n\beta_i\frac{\partial y^i}{\partial x^k}=\alpha_k\,,\qquad\sum_{i=1}^n\beta_i\frac{\partial y^i}{\partial\alpha_k}=0\,,\qquad k=1,\dots,n\,.$$

Since η is a diffeomorphism of class H^{s+1}, for fixed x, there exists a unique α such that $\beta=(0,\dots,1,\dots,0)$, the i-th basis vector. For this choice of α, the first relation shows that the i-th column of the matrix $\left(\dfrac{\partial y^i}{\partial x^k}\right)$ is H^{s+1}. This says that $y(x,\alpha)$ has all derivatives of order at most $s+2$ square integrable except the derivatives involving only α_k's. The second relation is an elliptic equation with H^{s+1} coefficients of first order in y^i regarded as a function of α only (its symbol maps $(\xi^i)\in\mathbb{R}^n$, to $(\beta_1+\dots+\xi^i\beta_i+\dots+\beta_n)\in\mathbb{R}^n$) and thus its solution is of class H^{s+2}, i.e. the $(s+2)nd$ derivative of y with respect to α is square integrable and thus y is of class H^{s+2}.

(2) Let $\eta\in\mathscr{D}_\theta^{s+1}(T^*M\backslash O)$ be fiber preserving, i.e. $\tau^*\eta(\alpha_x)=\tau^*\eta(\alpha'_x)$ for all α_x, $\alpha'_x\in T^*_xM\backslash O$. Then η can be extended H^{s+1}-smoothly to the zero section by $\eta(O_x)=O_y$ for $y=\tau^*\eta(\alpha_x)$, $\alpha_x\in T^*M\backslash O$. So $\eta:T^*M\to T^*M$, $\eta^*\theta=\theta$ and hence $\eta=T^*g$ for an H^{s+2} diffeomorphism $g:M\to M$. In particular if $\tau^*\eta(\alpha_x)=x$ for all $\alpha_x\in T^*M\backslash O$, then $\eta=e$. From this it follows that the effect of η on base points uniquely determines η, i.e. if $\eta,\bar\eta\in\mathscr{D}_\theta^{s+1}(T^*M\backslash O)$ satisfy $\tau^*\circ\eta=\tau^*\circ\bar\eta$, then $\eta=\bar\eta$ (since $\tau^*(\bar\eta\circ\eta^{-1})(\alpha_x)=x$).

(3) For all the function spaces introduced in this section when the index $s+1$ or $s+2$ is missing we always mean C^∞ mappings.

4. Global Writing of Pseudodifferential Operators and a Local Section from $\mathscr{D}_\theta(T^*M\backslash O)$ into $(\mathrm{FIO}_{0,k})_*$

We start by recalling a few facts about phase functions of Lagrangian submanifolds as they are presented in Weinstein [22].

Let $\pi:B\to M$ be a submersion and $N_\pi=\{\alpha\in T^*B|\alpha(v)=0$ for all $v\in\mathrm{Ker}(T\pi)\subset TB\}\subset T^*B$, the *conormal bundle to the fibers.* Locally, if q, (q,q'), $(q,q',\dot q,\dot q')$, (q,q',p,p') are coordinates on M, B, TB, T^*B respectively, and $\pi(q,q')=q$, then

$(q, q', O, \dot{q}')$ represents $\mathrm{Ker}(T\pi)$, and (q, q', p, O) is N_π. Thus the ω-orthogonal bundle $(TN_\pi)^\perp$ in TT^*B is generated by the vector fields $\partial/\partial q'$ and it follows that $(TN_\pi)^\perp \subset TN_\pi$, i.e. N_π is coisotropic, thus defining by the Frobenius theorem an integrable foliation, denoted by $\mathscr{N}_\omega^\perp$, of N_π by isotropic submanifolds. If $\dim B = n+k$, $\dim M = n$, the leaf of $\mathscr{N}_\pi^\perp$ through $\alpha_x = (q, q', p, O) \in N_\pi \subset T^*B$ is given locally by $(q, \mathbb{R}^k, p, O)$ and globally by $\{\beta_y \in N_\pi | \pi(y) = \pi(x)$, and if $T_x\pi(v_x) = T_y\pi(w_y)$, then $\alpha_x(v_x) = \beta_y(w_y)\}$. The quotient space $N_\pi/\mathscr{N}_\pi^\perp$, *the reduction of* T^*B *by* N_π, is locally a symplectic manifold with coordinates (q, O, p, O) and symplectic structure canonically induced from T^*B.

There exists a canonical map $\zeta : N_\pi/\mathscr{N}_\pi^\perp \to T^*M$, $\zeta(q, O, p, O) = (q, p)$ which is a local symplectic diffeomorphism. ζ is injective if and only if the fibers of π are connected and is surjective if and only if π is surjective. Thus if these conditions on π are satisfied $N_\pi/\mathscr{N}_\pi^\perp$ is a symplectic manifold symplectically diffeomorphic by ζ to T^*M. We assume this to be the case.

A C^2 map $\varphi : B \to \mathbb{R}$ is called a *global phase function* (or *Morse family*) *on* B, if the submanifold $d\varphi(B) \subset T^*B$ is transversal to N_π. In the above local coordinates, if $d\varphi(q, q') = (q, q', d_1\varphi(q, q'), d_2\varphi(q, q'))$ this means that the $k \times (n+k)$ matrix $(d_1 d_2 \varphi(q, q'), d_2 d_2 \varphi(q, q'))$ has rank k. From general theorems on reduction [22, 1, p. 417] it follows then that the reduced manifold $d\varphi(B)_{N_\pi} = (d\varphi(B) \cap N_\pi)/\mathscr{N}_\pi^\perp$ is Lagrangian in $N_\pi/\mathscr{N}_\pi^\perp \cong T^*M$. In the terminology of Fourier integral operators φ is indeed a global phase function for $d\varphi(B)_{N_\pi}$ as defined in Sect. 2. It is well known that every Lagrangian submanifold in T^*M admits *local* phase functions [8, 10, 22].

For the chart construction that follows one more ingredient is needed.

A *linearization* (Bokobza [4]) on a manifold M is a smooth map $v : \Omega \subset M \times M \to TM$, where Ω is an open neighborhood of the diagonal in $M \times M$, such that

1) $v(x, y) \in T_xM$, for all $(x, y) \in \Omega$
2) $v(x, x) = O_x \in T_xM$, for all $x \in M$
3) for all $x \in M$ the tangent map of $y \mapsto v(x, y)$, which is a linear map from T_yM to T_xM, is the identity for $y = x$.

Note that a linearization always exists by choosing a Riemannian metric and putting $v = \mathrm{Exp}^{-1}$, where $\mathrm{Exp} = (\tau, \exp)$: $TM \to M \times M$ is a diffeomorphism from a neighborhood U of the zero section of TM onto a neighborhood Ω of the diagonal in $M \times M$.

Now let in the prior construction $B = T^*M \times M$, and $\pi = \tau^* \times e : T^*M \times M \to M \times M$, i.e. $\pi(\alpha_x, y) = (x, y)$ and let $v : M \times M \to TM$ be a linearization of M.

Proposition 4.1. *The function* $\varphi_0 : T^*M \times M \to \mathbb{R}$ *defined by*

$$\varphi_0(\alpha_x, y) = \alpha_x \cdot (v(x, y))$$

is s smooth global phase function for graph(e) in $T^*M \times T^*M$.

Proof. B is isomorphic to the pullback bundle $p_1^*(T^*M)$, where $p_1 : M \times M \to M$ is the first projection, and $T\pi = T\tau^* \times \mathrm{id}$. Hence $\ker(T\pi) = (\mathrm{Ker}(T\tau^*)) \times O_{TM} = V_{TT^*M} \times O_{TM} \subset TT^*M \times TM$, where O_{TM} is the zero section in TM and V_{TT^*M} is the vertical subbundle of TT^*M. Furthermore, $N_\pi = H_{T^*T^*M} \times T^*M \subset T^*T^*M \times T^*M$, where the horizontal subbundle $H_{T^*T^*M} = \{\Gamma_{\alpha_x} \in T^*T^*M | \Gamma_{\alpha_x}(V_{TT^*M}) = 0\}$

is the annihilator of V_{TT^*M}. In local coordinates:

$$\ker(T\pi) = \{(x, \alpha, O, A; y, O) \quad \text{and} \quad N_\pi = \{(x, \alpha, \xi, O; y, \eta)\}.$$

Recall that $v(x, y)$ is a diffeomorphism from a neighborhood Ω of Δ in $M \times M$ onto a neighborhood U of O_{TM} in TM. So locally $v(x, y) = (x, v'(x, y))$ where $v'(x, y) = 0$ if and only if $x = y$. We are working in the open submanifold $\pi^{-1}(\Omega)$ in $T^*M \times M$. Note that this submanifold contains whole fibers of T^*M.

To show that φ_0 is a Morse family we work in local coordinates where $\varphi_0(x, \alpha; y) = \alpha(v'(x, y))$ and straightforward calculations show that

$$d\varphi_0(\alpha_x, y) = \{((x, \alpha, \alpha \circ d_1 v'(x, y), v'(x, y)), (y, \alpha \circ d_2 v'(x, y))\} \in T^*_{(\alpha_x, y)}(T^*M \times M)$$

and that $T_{d\varphi_0(\alpha_x, y)}(N_\pi)$ is of the form

$$T_{d\varphi_0(\alpha_x, y)}(N_\pi) = \{(X, A, W, O; Y, b)\} \subset T_{d\varphi_0(\alpha_x, y)}(T^*T^*M \times T^*M).$$

Furthermore $\operatorname{Im}(T_{(\alpha_x, y)} d\varphi_0)$ is locally of the form (put $V_{\alpha_x} = (x, \alpha, X, A)$, $Y_y = (y, Y)$)

$$(T_{(\alpha_x, y)} d\varphi_0)(V_{\alpha_x}, Y_y) = \{(X, A, \alpha \circ (d_1^2 v'(x, y) \cdot X + d_2 d_1 v'(x, y) \cdot Y), d_1 v'(x, y) \cdot X + d_2 v'(x, y) \cdot Y, (Y, \alpha \circ (d_1 d_2 v'(x, y) \cdot X + d_2^2 v'(x, y) \cdot Y))\}.$$

By property 3) of v, this shows that for all $(\alpha_x, y) \in T^*M \times M$ such that $d\varphi_0(\alpha_x, y) \in N_\pi$ we have

$$\operatorname{Im}(T_{(\alpha_x, y)} d\varphi_0)(V_{\alpha_x}, Y_y) + T_{d\varphi_0(\alpha_x, y)}(N_\pi) = T_{d\varphi_0(\alpha_x, y)}(T^*T^*M \times T^*M)$$

i.e. $d\varphi_0(T^*M \times M)$ is transversal to N_π.

The induced foliation $\mathcal{N}_\pi^\perp$ is of the form $\mathcal{N}_\pi^\perp = \{(x, \mathbb{R}^n, \xi, O; y, \eta)\}$ and hence $N_\pi/\mathcal{N}_\pi^\perp = \{(x, y, \xi, \eta)\} = T^*(M \times M)$. So $d\varphi_0(T^*M \times M)_{N_\pi} = (d\varphi_0(T^*M \times M) \cap N_\pi)/\mathcal{N}_\pi^\perp = \{(x, x, \alpha \circ d_1 v'(x, x), \alpha)\} = \{x, x, -\alpha, \alpha)\}$ since $v'(x, y) = 0 \Leftrightarrow x = y$ and $d_1 v'(x, x) = -\mathrm{id}$. $\square$

Remarks 1). Take $B = (T^*M \backslash O) \times M$ and φ_0 as in Proposition 4.1. The proof shows that

$$d\varphi_0((T^*M \backslash O) \times M)_{N_\pi} = \{(x, x, -\alpha, \alpha) | \alpha \neq 0\}$$

$$\text{i.e. } d\varphi_0((T^*M \backslash O) \times M)_{N_\pi} = \operatorname{graph}(e) \subset (T^*M \backslash O) \times (T^*M \backslash O), \omega \ominus \omega).$$

2) $\pi : (T^*M \backslash O) \times M \to M \times M$ has not connected fibers if and only if $\dim M = 1$. But in this case we have $M = S^1$ and $T^*M \backslash O = (\mathbb{R} \backslash O) \times S^1$ and the reduced manifold is $d\varphi_0((T^*M \backslash O) \times M)_{N_\pi} = \{(x, x, -\alpha, \alpha) | \alpha > 0\} \cup \{(x, x, -\alpha, \alpha) | \alpha < 0\}$ which is a Lagrangian submanifold in

$$N_\pi/\mathcal{N}_\pi^\perp = (S^1 \times \mathbb{R} \times S^1 \times \mathbb{R})_+ \cup (S^1 \times \mathbb{R} \times S^1 \times \mathbb{R})_-.$$

In terms of a linearization on M we are able to write a global formula for pseudodifferential operators on a manifold, [4].

Let $v : \Omega \subset M \times M \to TM$ be a linearization of M and let $U \subset TM$ be the neighborhood of the zero section in TM where $v : \Omega \to U$ is a diffeomorphism. For any $x \in M$ let $\Omega_x = \{y \in M | (x, y) \in \Omega\}$ and $U_x = U \cap T_x M$, then $v_x = v(x, \cdot) : \Omega_x \to U_x$ is a diffeomorphism. The tangent map $T_y v_x : T_y M \to T_{v(x,y)} T_x M \cong T_x M$ is the identity for $y = x$. Its dual $T_y^* v_x : T_x^* M \to T_y^* M$ induces a linear map of densities

$$|\Lambda^n(T_y^* v_x)| : |\Lambda^n T_x^* M| \to |\Lambda^n T_y^* M| \quad \text{for} \quad y \in \Omega_x.$$

Choose a density ζ_x on T_xM, i.e. $\zeta_x \in |\Lambda^n T_x^* M|$ and pull it back to Ω_x by means of $|\Lambda^n T_y^* v_x|$, i.e.

$$(v_x^* \zeta_x)(y) = |\Lambda^n T_y^* v_x|(\zeta_x) \in |\Lambda^n T_y^* \Omega_x| = |\Lambda^n T_y^* M|$$

for $y \in \Omega_x$. In local coordinates $\partial_{x_1}, \ldots, \partial_{x_n}$ on T_xM, and $y_1, \ldots, y_n$ on Ω_x we can choose $\zeta_x = dx = dx_1 \wedge \ldots \wedge dx_n$ and $v_v^* \zeta_x = v_x^* dx$

$$= |\det_{(dx,dy)} v_x| dy_1 \wedge \ldots \wedge dy_n = \det \left| \frac{\partial v(x,y)}{\partial y} \right| dy .$$

Now, let $d\xi \in |\Lambda^n T_x M|$ be the dual basis of $\zeta_x \in |\Lambda^n T_x^* M|$. Considering $d\xi$ as a density on $T_x^* M$ we have $d\xi \otimes v_x^* d\zeta_x$ is a density on $T_x^* M \times \Omega_x$.

Let $f: T^*M \times M \to \mathbb{C}$, $f(\alpha_x, y)$ be such that $\operatorname{supp}_y f \subset \Omega_x$ for all $\alpha_x \in T^*M$. Then the integral

$$\int_{\Omega_x} f(\alpha_x, y) |\det_{(dx,dy)} v_x| dy$$

is a function on $T_x^* M$ which we can integrate with respect to $d\xi$,

$$\int_{T_x^* M} d\xi \int_{\Omega_x} f(\alpha_x, y) |\det_{(dx,dy)} v_x| dy \in \mathbb{C} .$$

Let $\chi(x,y)$ be a bump function on $M \times M$ such that $\operatorname{supp} \chi \subset \Omega$ and $\chi \equiv 1$ on a neighborhood of the diagonal. Let $a(x,\xi) \in S^m(\Omega)$ be a classical symbol of order m. We define an operator P on M by

$$Pu(x) = (2\pi)^{-n} \int_{T_x^* M} a(x,\xi) d\xi \int_{\Omega_x} \chi(x,y) e^{i\varphi_0(\alpha_x, y)} u(y) |\det_{(dx,dy)} v_x| dy , \qquad (4.1)$$

where $u \in C^\infty(M)$ and φ_0 is the global phase function of the diagonal in $T^*M \times T^*M$ defined in Proposition 4.1. The integral (4.1) converges in the oscillatory sense.

Bokobza [4] showed that P is a continuous linear operator from $C^\infty(M)$ to $C^\infty(M)$, and from $\mathscr{D}(M)$ into $\mathscr{D}(M)$, i.e. P is a classical pseudodifferential operator of order m on M in our previous sense.

Remark. If $M = \mathbb{R}^n$, a linearization $v: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is given by $v(x,y) = x - y$, and the phase function $\varphi_0: (\mathbb{R}^n)^* \times \mathbb{R}^n \to \mathbb{R}$ of graph(e), becomes $\varphi_0(\xi, y) = \xi \cdot (x - y)$. So a pseudodifferential operator P on $\mathbb{R}^n$, according to (4.1) can be written as

$$Pu(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a(x,\xi) u(y) dy d\xi \qquad (4.2)$$

which is indeed the standard form (2.3).

Now by perturbing φ_0 by "small" functions $H \in \mathscr{S}^{s+2}(T^*M \backslash O)$ we obtain global phase functions for Lagrangian submanifolds close to graph(e) in $((T^*M\backslash O) \times (T^*M \backslash O), \omega \ominus \omega)$.

Lemma 4.2. *Let $H \in \mathscr{S}_W^{s+2}(T^*M\backslash O)$ be close to zero and define $\varphi_H: (T^*M\backslash O) \times M \to \mathbb{R}$ by*

$$\varphi_H(\alpha_x, y) = \varphi_0(\alpha_x, y) + H(\alpha_x) .$$

Then there exists an $\eta \in \mathscr{D}_\theta^{s+1}(T^*M\backslash O)$ such that φ_H is a global phase function for graph(η), *i.e.* $d\varphi_H((T^*M\backslash O) \times M)_{N_\pi} = \operatorname{graph}(\eta)$.

Proof. The Lagrangian submanifold in $(T^*M\backslash O)\times(T^*M\backslash O)$ generated by φ_H is locally

$$d\varphi_H((T^*M\backslash O)\times M)_{N_\pi}=\{(x,y,\alpha\circ d_1v'(x,y)+d_1H(x,\alpha),\alpha\circ d_2v'(x,y))|v'(x,y)+d_2H(x,\alpha)=0\}.$$

So we have to solve the system of equations for (x,α) when (y,β) are given

$$\alpha\circ d_2v'(x,y)=-\beta, \tag{4.3}$$

$$v'(x,y)+d_2H(x,\alpha)=0. \tag{4.4}$$

Since we are looking for solutions (x,α) only in the neighborhood Ω of Δ in $M\times M$ where v is a diffeomorphism, we can solve (4.3) for α, $\alpha=-\beta\circ d_2v'(x,y)^{-1}$. The map $\phi_y=M\to TM$ defined for every $(x,y)\in\Omega$ by $\phi_y(x)=v'(x,y)+d_2H(x,-\beta\circ d_2v'(x,y)^{-1})$ is for $H\equiv 0$ transversal to O_{TM} in TM and $y\in\phi_y^{-1}(O_{TM})$. Since $H\in\mathscr{S}_W^{s+2}(T^*M\backslash O)$ is close to zero, i.e. dH is C^1-close to zero, ϕ_y is still transversal to O_{TM} and $\phi_y^{-1}(O_{TM})\neq\emptyset$. By the implicit function theorem there is a unique $x\in\Omega_y$ such that $\phi_y(x)=0$. Denote this $x(y,\beta)$ and let $\xi(y,\beta)=d_1v'(x(y,\beta),y)+d_1H(x(y,\beta),\alpha)$. This defines a diffeomorphism $\eta:(T^*M\backslash O)\to(T^*M\backslash O)$ such that $\eta^{-1}(y,\beta)=(x(y,\beta),\xi(y,\beta))$ and (4.4) is satisfied. So $d\varphi_H((T^*M\backslash O)\times M)_{N_\pi}=\operatorname{graph}(\eta)$. Since $d\varphi_H((T^*M\backslash O)\times M)_{N_\pi}$ is a conic Lagrangian submanifold it follows that $\eta^*\theta=\theta$ i.e. $\eta\in\mathscr{D}_\theta^{s+1}(T^*M\backslash O)$. □

Lemma 4.3. *The map $H\mapsto\eta$ given by Lemma 4.2 is a bijection from a neighborhood $\mathscr{V}$ of zero in $\mathscr{S}_W^{s+2}(T^*M\backslash O)$ onto a neighborhood $\mathscr{U}$ of the identity in $\mathscr{D}_\theta^{s+1}(T^*M\backslash O)$. The inverse mapping is given as follows: for $\eta\in\mathscr{D}_\theta^{s+1}(T^*M\backslash O)$ define $H\in\mathscr{S}^{s+2}(T^*M\backslash O)$ by $H(\alpha_x)=-\alpha_x\cdot v(x,\tau^*\eta^{-1}(\alpha_x))$.*

Proof. Given $\eta\in\mathscr{D}_\theta^{s+1}(T^*M\backslash O)$, associate to it $H\in\mathscr{S}^{s+2}(T^*M\backslash O)$ by

$$H(\beta_z)=-\beta_z\cdot v(z,\tau^*\eta^{-1}(\beta_z)). \tag{4.5}$$

H is H^{s+2} because $\tau^*\circ\eta^{-1}$ is H^{s+2} (Remark 1 at the end of Sect. 3) and it is homogeneous of degree one. With this we build the phase function φ_H by Lemma 4.2

$$\varphi_H(\beta_z,y)=\varphi_0(\beta_z,y)+H(\beta_z)=\beta_z\cdot v(z,y)-\beta_z\cdot v(z,\tau^*\eta^{-1}(\beta_z)).$$

The Langrangian submanifold in $((T^*M\backslash O)\times(T^*M\backslash O),\omega\ominus\omega)$ generated by φ_H is

$$d\varphi_H((T^*M\backslash O)\times M)_{N_\pi}=\{(x,z,\gamma\circ d_1v'(x,z)+d_1H(x,\gamma),\gamma\circ d_2v'(x,z))|d_2H(x,\gamma)=-v'(x,z)\}.$$

Define $h:(T^*M\backslash O)\to(T^*M\backslash O)$ by $h(\beta_z)=-\beta_z\circ T_2v(\tau^*\eta(\beta_z),z)^{-1}$, where $T_2v(x,y)=T_yv(x,\cdot):T_yM\to T_xM$. If η is close to $e\in\mathscr{D}_\theta^{s+1}(T^*M\backslash O)$ we get $(\tau^*\eta(\beta_z),z)\in\Omega$ where Ω is the neighborhood of $\Delta\subset M\times M$ where v is a diffeomorphism. It is easy to check that h is an H^{s+2} diffeomorphism homogeneous of degree one with inverse $h^{-1}(\alpha_x)=\alpha_x\circ T_2v(x,\tau^*\eta^{-1}(\alpha_x))$.

Take $H\in\mathscr{V}\subset\mathscr{S}_W^{s+2}(T^*M\backslash O)$ and associate to it $\eta\in\mathscr{U}\subset\mathscr{D}_\theta^{s+1}(T^*M\backslash O)$ according to Lemma 4.2. This defines an $\bar{H}\in\mathscr{S}^{s+2}(T^*M\backslash O)$ by (4.5). To show that $H=\bar{H}$ it is sufficient to prove $H\circ h=\bar{H}\circ h$. Using the fact that H is homogeneous of degree one and (4.4) we get

$$\begin{aligned}(\bar{H}\circ h)(z,\beta) &= \bar{H}(x,-\beta\circ d_2v'(x,z)^{-1}) = \beta\circ d_2v'(x,z)^{-1}\cdot v'(x,z)\\ &= -\beta\circ d_2v'(x,z)^{-1}\cdot d_2H(x,-\beta\circ d_2v'(x,z)^{-1})\\ &= H(x,-\beta\circ d_2v'(x,z)^{-1})\\ &= (H\circ h)(z,\beta).\end{aligned}$$

Given $\eta\in\mathcal{U}\subset\mathcal{D}_\theta^{s+1}(T^*M\backslash O)$ define H by (4.5) and associate to it $\bar{\eta}$ according to Lemma 4.2, i.e. $\bar{\eta}$ is defined from $d\varphi_H((T^*M\backslash O)\times M)_{N_\pi}$. Given (z,β) with $\beta=\gamma\circ d_2v'(x',z)$ where $x'=\tau^*\bar{\eta}(z,\beta)$ is the unique solution of

$$d_2H(x',\gamma) = -v'(x',z). \tag{4.6}$$

Let $\eta^{-1}(z,\beta)=(a(z,\beta),\,b(z,\beta))$. Then $H(z,\beta)=-\beta\cdot v'(z,a(z,\beta))$ and $d_2H(z,\beta)=-v'(z,a(z,\beta))-\beta\circ d_2v'(z,a(z,\beta))\circ d_2a(z,\beta)$.
Therefore (4.6) becomes

$$-v'(x',a(x',\gamma))-\gamma\circ d_2v'(x',a(x',\gamma))\circ d_2a(x',\gamma) = -v'(x',z). \tag{4.6'}$$

Now let $x=\tau^*\eta(\beta_z)$, and replace in (4.6′) everywhere x' by x. Since $a(x,\gamma)=a(h(z,\beta))=z$ and $\gamma\circ d_2v'(x,a(x,\gamma))=\beta\circ d_2v'(x,z)^{-1}\circ d_2v'(x,z)=\beta$, (4.6′) becomes

$$\beta\circ d_2a(x,\beta\circ d_2v'(x,z)^{-1})=0. \tag{4.6''}$$

Since $h(z,\beta)=(x,\gamma)$, calling $h^{-1}(x,\gamma)=(a(x,\gamma),c(x,\gamma))$ it follows $c(x,\gamma)=\beta$ and (4.6″) becomes $c(x,\gamma)\circ d_2a(x,\gamma)=0$ which is verified since h^{-1} is homogeneous of degree one and $(h^{-1})^*\theta(x,\gamma)(A,B)=(c\circ d_1a)\cdot A+(c\circ d_2a)\cdot B=\theta(x,\gamma)(A,B)=\gamma\cdot A$ for any vector (A,B). Hence $x=\tau^*\eta(\beta_z)$ solves (4.6) follows $\tau^*\circ\bar{\eta}=\tau^*\circ\eta$ and by Remark 2 at the end of Sect. 3, we get $\bar{\eta}=\eta$. □

Twice application of the Ω-lemma [16] proves that $\eta^{-1}\mapsto H$ is a C^∞ map. Since the map $H\mapsto\eta^{-1}$ is continuous, bijective and has a C^∞ inverse, it is a C^∞ diffeomorphism.

Since $\mathcal{S}^{s+2}(T^*M\backslash O)$ is continuously imbedded in $\mathcal{S}_W^{s+2}(T^*M\backslash O)$ the prior lemmas yield the following theorem.

Theorem 4.4. *$(\mathcal{U},\Phi,\mathcal{S}^{s+2}(T^*M\backslash O))$ is a smooth chart at the identity of $\mathcal{D}_\theta^{s+1}(T^*M\backslash O)$, where $\Phi:\mathcal{U}\subset\mathcal{D}_\theta^{s+1}(T^*M\backslash O)\to\mathcal{S}^{s+2}(T^*M\backslash O)$ is defined by*

$$\Phi(\eta)(\alpha_x) = -\alpha_x\cdot v(x,\tau^*\eta^{-1}(\alpha_x))$$

for all $\alpha_x\in T^*M\backslash O$.

Remarks. (1) This chart depends upon the linearization v. But two different linearizations v and $\tilde{v}$ define a diffeomorphism

$$g:(T^*M\backslash O)\times M\to(T^*M\backslash O)\times M,\ \pi\circ g=\pi,\ \text{satisfying } \tilde{\varphi}_0=\varphi_0\circ g$$

and therefore they generate the same Lagrange submanifold. This is crucial for our application to Fourier integral operators because these are invariant under change of phase functions as long as they define the same Lagrangian submanifold.

(2) Note that $\eta\in\mathcal{U}$ is C^∞ if and only if $\Phi(\eta)$ is C^∞.

We are now equipped to generalize the global formula (4.1) for pseudodifferential operators on a manifold to give a global formula for Fourier integral operators near the identity. Let $A \in \mathrm{FIO}_m(\eta)$ with $\eta \in \mathscr{U} \cap \mathscr{D}_\theta(T^*M\backslash O)$ and let $a(x,\xi)$ be a representative of the classical symbol of A. Let $\chi(x,y)$ be the bump function used in (4.1), then, modulo smoothing,

$$Au(x) = (2\pi)^{-n} \int_{T_x^*M} a(x,\xi)\,d\xi \int_{\Omega_x} \chi(x,y) e^{i\varphi_H(\alpha_x,y)} u(y)\,|\det{}_{(dx,dy)} v_x|\,dy\,, \tag{4.7}$$

where φ_H is the global phase function of graph(η), given by Lemmas 4.2 and 4.3.

Remark. Let $M = \mathbb{R}^n$ and take the linearization $v(x,y) = x - y$. Then for $H \in \Phi(\mathscr{U}) \cap \mathscr{S}(\mathbb{R}^{2n}\backslash O)$ we get $\varphi_H(\xi,y) = (x-y)\cdot\xi + H(\xi)$. Thus a representative of a formal Fourier integral operator A on $\mathbb{R}^n$ can be written, according to (4.7), as

$$Au(x) = (2\pi)^{-n} \iint e^{i((x-y)\cdot\xi + H(\xi_x))} a(x,\xi) u(y)\,dy\,d\xi \tag{4.8}$$

which corresponds to the standard form [5, 10].

Finally, we can define a local section σ from $\mathscr{D}_\theta(T^*M\backslash O)$ into $(\mathrm{FIO}_{0,k})_*$ as follows:

For $\eta \in \mathscr{U} \cap \mathscr{D}_\theta(T^*M\backslash O)$, define $\sigma(\eta) \in (\mathrm{FIO}_{0,k})_*$ by

$$\sigma(\eta)u(x) = (2\pi)^{-n} \int_{T_x^*M} d\xi \int_{\Omega_x} \chi(x,y) e^{i\varphi_H(\alpha_x,y)} u(y)\,|\det{}_{(dx,dy)} v_x|\,dy\,, \tag{4.9}$$

where $\varphi_H(\alpha_x, y) = \varphi_0(\alpha_x, y) + H(\alpha_x)$ and $H = \Phi(\eta)$, so $\varphi_H(\alpha_x, y) = \alpha_x \cdot (v(x,y)) - \alpha_x \cdot v(x, \tau^*\eta^{-1}(\alpha_x))$. Therefore

$$\sigma(\eta)u(x) = (2\pi)^{-n} \int_{T_x^*M} d\xi \int_{\Omega_x} \chi(x,y) e^{i\alpha_x\cdot(v(x,y) - v(x,\tau^*\eta^{-1}(\alpha_x)))} u(y) \left|\det \frac{\partial v(x,y)}{\partial y}\right| dy\,.$$

In other words, $\sigma(\eta)$ is defined as the Fourier integral operator whose phase function is $\varphi_{\Phi(\eta)}$, the global generating function of graph(η), and whose amplitude is 1. Recall that $\Phi(\eta)$ is smooth iff η is smooth, in which case $\sigma(\eta)$ is a well defined Fourier integral operator of order zero. Its principal symbol is 1 (in the trivializations given by the global phase function $\varphi_{\Phi(\eta)}$), hence $\sigma(\eta)$ is invertible modulo smoothing operators, in particular, $\sigma(\eta) \in (\mathrm{FIO}_{0,k})_*$ for any k. Notice $\pi(\sigma(\eta)) = \eta$ for all $\eta \in \mathscr{U} \cap \mathscr{D}_\theta(T^*M\backslash O)$, i.e. σ is a local section.

5. The Lie Group Structure of $(\psi DO_{0,k})_*$

The space $(\psi DO_0)_*$ can be given the structure of a topological group as follows. Let v be a linearization of M and $\varphi_0 : T^*M \times M \to \mathbb{R}$ the global phase function given by v as in Proposition 4.1. Using φ_0 we can write a representative of $[P] \in \psi DO_m$ in the form (4.1) where $a(x,\xi) \sim \sum_{j=0}^{\infty} a_{m-j}(x,\xi)$, $a_{m-j}(x,\xi)$ being a smooth function on $T^*M\backslash O$, homogeneous of degree $m-j$. To do this let $a_m(x,\xi)$ be the principal symbol of $[P] \in \psi DO_m$ and let P_m be the operator given by (4.1) with total symbol $a_m(x,\xi)$, i.e. replace $a(x,\xi)$ in (4.1) by $a_m(x,\xi)$. Then $[P - P_m] \in \psi DO_{m-1}$. Let $a_{m-1}(x,\xi)$ be the principal symbol of $[P - P_m]$ and let P_{m-1} be given by (4.1) with total symbol $a_{m-1}(x,\xi)$, then $[P - P_m - P_{m-1}] \in \psi DO_{m-2}$. Continue in this fashion to get $a_{m-j}(x, \xi)$ and P_{m-j} so that $[P-P_m-P_{m-1}-\ldots-P_{m-j}] \in \psi DO_{m-j-1}$ for all $j \geqq 0$. The sequence $(a_m(x,\xi), a_{m-1}(x,\xi), \ldots)$ will be called the *symbol* of $[P]$ with respect to the phase function φ_0. Clearly, in this way, we have constructed an isomorphism between ψDO_m and the space of sequences $\{(a_m(x,\xi), a_{m-1}(x,\xi), \ldots) | a_{m-j}(x,\xi) \in C^\infty(T^*M\backslash O)$ and is homogeneous of degree $m-j\}$. This can be identified to a somewhat simpler space by restricting to a cosphere bundle. Namely, as in Sect. 3 let $Q = T^*M\backslash O/\mathbb{R}^+$ be the cosphere bundle and choose $\sigma: Q \to T^*M\backslash O$ a section of the $\mathbb{R}^+$ bundle $\pi: T^*M\backslash O \to Q$. Then by pulling back functions on $T^*M\backslash O$ to Q by σ we can identify ψDO_m with the infinite product $C^\infty(Q) \times C^\infty(Q) \times \ldots$ by $[P] \mapsto (a_m \circ \sigma, a_{m-1} \circ \sigma, \ldots)$ where $(a_m, a_{m-1}, \ldots)$ is the symbol of $[P]$. Since the product $C^\infty(Q) \times C^\infty(Q) \times \ldots$ is a topological space we can endow ψDO_m with the topology induced by this identification. It can be seen that the topology is independent of the choices of v and σ and that composition on ψDO_m is continuous in this topology (c.f. Propositions 5.3, 5.4, and 5.5). From this last remark it follows that $(\psi DO_0)_*$ is a topological group. The problem with this topological group arises when we try to find the spaces ψDO_0^s to make $(\psi DO_0)_*$ an ILH group. One would want to take an infinite product of various H^s spaces on Q. Unfortunately the infinite product of Hilbert spaces is no longer a Hilbert space. For this reason we are forced to use the spaces $\psi DO_{m,k}$ and then consider ψDO_m as a direct limit of the ILH spaces $\psi DO_{m,k}$ as $k \to \infty$.

Following the construction above we choose a global phase function φ_0, coming from the linearization v, and a section $\sigma: Q \to T^*M\backslash O$. Then $\psi DO_{m,k}$ is isomorphic to the $(m+k+1)$-fold product $C^\infty(Q) \times \ldots \times C^\infty(Q)$ by extending the jth function to $T^*M\backslash O$ by homogeneity of degree $m-j$, $0 \leqq j \leqq m+k$, and then using φ_0 to give an identification of $\psi DO_{m,k}$ with the finite sequences $(a_m(x,\xi), a_{m-1}(x,\xi), \ldots, a_{m-k}(x,\xi))$. We will see momentarily that the induced topology on $\psi DO_{m,k}$ is independent of the choices v and σ and that $(\psi DO_{0,k})_*$ is a topological group with this topology. First we describe the spaces $\psi DO_{m,k}^s$.

We use H^s structures on the symbol spaces to define $\psi DO_{m,k}^s$. Let $a_l(x,\xi)$ be a smooth function on $T^*M\backslash O$ which is homogeneous of degree l in ξ. If $l \geqq -n$, $a_l(x,\xi)$ won't be in any of the spaces $H^s(T^*M\backslash O)$ because of its growth at infinity. To get around this we restrict to the cosphere bundle Q by means of σ. Namely, define (for $s > \dim M$)

$$\|a_l(x,\xi)\|_s = \|a_l \circ \sigma\|_s, \tag{5.1}$$

where the norm $\|\cdot\|_s$ on the right is the H^s norm of $a_l \circ \sigma$ as a function on the compact manifold Q. Let $\mathscr{S}_l^s(T^*M\backslash O)$ be the completion of the set of smooth functions on $T^*M\backslash O$, homogeneous of degree l, with respect to this topology. This space can of course be thought of as the space of H^s functions on $S^*M = \sigma(Q)$, extended to $T^*M\backslash O$ by homogeneity of degree l.

Before proceeding, we gather some of the properties of these norms.

Proposition 5.1. *The Hilbert manifold structure of $\mathscr{S}_l^s(T^*M\backslash O)$ is independent of the choice of section $\sigma: Q \to T^*M\backslash O$.*

Proof. Let $\tilde{\sigma}: Q \to T^*M\backslash O$ be another section. As in Sect. 3 there is a function f_σ such that

$$\sigma(\pi(\alpha_x)) = f_\sigma(\alpha_x)\alpha_x, \quad \forall \alpha_x \in T^*M\backslash O.$$

Let $h_{\sigma\tilde{\sigma}}: Q \to \mathbb{R}$ be given by

$$h_{\sigma\tilde{\sigma}} = f_\sigma \circ \tilde{\sigma}.$$

$h_{\sigma\tilde{\sigma}}$ tells us how far (multiplicatively) $\tilde{\sigma}(q)$ is from $\sigma(q)$, namely $\sigma(q) = h_{\sigma\tilde{\sigma}}(q)\tilde{\sigma}(q)$. Hence if $a \in \mathscr{S}_l^s(T^*M\backslash O)$ then

$$(a \circ \sigma)(q) = (h_{\sigma\tilde{\sigma}}(q))^l (a \circ \tilde{\sigma})(q).$$

But $h_{\sigma\tilde{\sigma}}$ is a fixed smooth, nonvanishing function on Q so that multiplication by $(h_{\sigma\tilde{\sigma}})^l$ is a diffeomorphism on $H^s(Q)$. $\square$

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^n$ be open and let $a(x,\xi) \in \mathscr{S}_l^s(T^*\Omega\backslash O$. Then*

$$D_x^\alpha D_\xi^\beta a(x,\xi) \in \mathscr{S}_{l-|\beta|}^{s-|\alpha|-|\beta|}(T^*\Omega\backslash O).$$

Proof. By Proposition 5.1 we are free to choose a convenient section $\sigma: Q \to T^*\Omega\backslash O$. We choose σ to give the unit sphere bundle in the standard metric on $\mathbb{R}^n$, i.e. $\sigma(Q) = \{(x,\xi) | x \in \Omega, |\xi| = 1\}$.

We need to bound the norm of $D_x^\alpha D_\xi^\beta (a \circ \sigma)$ in $H^{s-|\alpha|-|\beta|}(Q)$ by the norm of $a \circ \sigma$ in $H^s(Q)$. The x-derivatives are no problem so it is enough to prove the statement for $|\alpha| = 0$, $|\beta| = 1$ (the argument can be repeated for longer β). To do this write the operator $\frac{\partial}{\partial \xi_i}$ as the sum of a radial part and a tangential part, $\frac{\partial}{\partial \xi_i} = D_{T_i} + D_{R_i}$. In this case $D_{R_i} = \frac{\xi_i}{|\xi|}\frac{\partial}{\partial r}$ where $r = |\xi|$. By the homogeneity of $a_l(x,\xi)$ we have that

$$|\xi| \frac{\partial}{\partial r} a_l(x,\xi) = (l-1)a(x,\xi)$$

so that

$$D_{R_i}(a \circ \sigma) = (\xi_i(l-1)a) \circ \sigma.$$

Now ξ_i is a smooth function on Q, bounded by 1, so

$$\|(\xi_i(l-1)a) \circ \sigma\|_0 \leqq (l-1)\, \|a \circ \sigma\|_0.$$

It follows that

$$\|D_{\xi_i}(a \circ \sigma)\|_{s-1} \leqq C\, \|a \circ \sigma\|_s \quad \text{for some } C. \quad \square$$

We are now ready to define the s-norm of an element $[P] \in \psi DO_{m,k}$. Fix a linearization v on M and let $\sum_{j=0}^{m+k} a_{m-j}$ be the symbol of P with respect to this linearization. Define

$$\|P\|_{m,k;s}^2 = \|a_m\|_{s+k+m}^2 + \|a_{m-1}\|_{s+k+m-1}^2 + \ldots + \|a_{-k}\|_s^2, \tag{5.2}$$

where $\|a_{m-j}\|_{s+m+k-j}$ is the norm of a_{m-j} in $\mathscr{S}_{m-j}^{s+k+m-j}(T^*M\backslash O)$ given by some choice of section $\sigma: Q \to T^*M\backslash O$. Let $\psi DO_{m,k}^s$ denote the completion of $\psi DO_{m,k}$ with respect to this norm. This makes $\psi DO_{m,k}^s$ a Hilbert space. As an immediate consequence of Proposition 5.1 we have

Proposition 5.3. *The Hilbert manifold structure of $\psi DO^s_{m,k}$ is independent of the choice of section $\sigma: Q \to T^*M \backslash O$.*

The norm also depends on the linearization v.

Proposition 5.4. *The Hilbert manifold structure on $\psi DO^s_{m,k}$ is independent of the choice of linearization.*

Proof. We work with a fixed section $\sigma: Q \to T^*M \backslash O$ and suppose $\tilde{v}$ is another linearization. Let φ_0 and $\tilde{\varphi}_0$ be the phase functions induced by v and $\tilde{v}$. Then as in (4.1) for $[P] \in \psi DO_{m,k}$ we can write

$$\begin{aligned} Pu(x) &= (2\pi)^{-n} \int_{T^*_xM} a(x,\xi)d\xi \int_{\Omega_x} \chi(x,y) e^{i\varphi_0(\alpha_x,y)} u(y) |\det v_x| dy \\ &= (2\pi)^{-n} \int_{T^*_xM} e^{i(\varphi_0(\alpha_x,y)-\tilde{\varphi}_0(\alpha_x,y))} \left|\frac{\det v_x}{\det \tilde{v}_x}\right| a(x,\xi)d\xi \int_{\Omega_x} \chi(x,y) e^{i\tilde{\varphi}_0(\alpha_x,y)} \\ &\quad \cdot u(y) |\det \tilde{v}_x| dy \\ &= (2\pi)^{-n} \int_{T^*_xM} \tilde{a}(x,\xi)d\xi \int_{\Omega_x} \chi(x,y) e^{i\tilde{\varphi}_0(\alpha_x,y)} u(y) |\det \tilde{v}_x| dy, \end{aligned}$$

where $\tilde{a}(x,\xi)$ is given locally by the asymptotic formula

$$\tilde{a}(x,\xi) = (2\pi)^n \sum_\alpha \frac{1}{\alpha!} (iD_\xi)^\alpha D_y^\alpha \tilde{a}(x,y,\xi)|_{y=x} \tag{5.3}$$

with

$$\tilde{a}(x,y,\xi) = a(x,\xi) e^{i(\varphi_0(x,y,\xi)-\tilde{\varphi}_0(x,y,\xi))} \left|\frac{\det v_x}{\det \tilde{v}_x}\right| (y)$$

(see e.g. [10]). From this we get that

$$\begin{aligned} \tilde{a}_m(x,\xi) &= a_m(x,\xi), \\ \tilde{a}_{m-1}(x,\xi) &= a_{m-1}(x,\xi) + \frac{1}{(-1)^{1/2}} \sum_{j=1}^n \frac{\partial}{\partial \xi_j} a_m(x,\xi) \frac{\partial}{\partial y_j} \left|\frac{\det v_x}{\det \tilde{v}_x}\right| (y)|_{y=x} \\ &\text{etc.} \end{aligned} \tag{5.4}$$

By Proposition 5.2 the derivatives $D_\xi^\alpha a_{m-j}(x,\xi)$ are bounded in the $H^{s+m+k-j-|\alpha|}$ norm by the $H^{s+m+k-j}$ norm of $a_{m-j}(x,\xi)$. Since also v and $\tilde{v}$ are smooth the map

$$\sum_{j=0}^{m+k-1} a_{m-j}(x,\xi) \mapsto \sum_{j=0}^{m+k-1} \tilde{a}_{m-j}(x,\xi)$$

is a diffeomorphism. $\square$

By choosing a linearization v and a section $\sigma: Q \to T^*M \backslash O$ we can think of $[P] \in \psi DO^s_{m,k}$ as represented by a pseudodifferential operator with symbol $a(x,\xi) = \sum_{j=0}^{m+k} a_{m-j}(x,\xi)$ with $a_{m-j} \circ \sigma \in H^{s+m+k-j}(Q)$. In fact this description can be given rigorous sense if s is large enough. Indeed, the operator defined by (4.1) still makes sense as an oscillatory integral if we can differentiate $a(x,\xi)$ enough times so that it becomes integrable on $T^*M \backslash O$. If $s > 2n+m$ the first $n+m$ derivatives of $a(x,\xi)$ are continuous so integration by parts $n+m$ times gives a convergent expression.

Recall that if $m=0$ composition in $\psi DO_{0,k}$ is well defined.

Proposition 5.5. *Composition extends continuously to* $\psi DO^s_{0,k}$.

Proof. If P_1 has symbol $a=a_0(x,\xi)+a_{-1}(x,\xi)+\ldots+a_{-k}(x,\xi)$ and P_2 has symbol $b=b_0(x,\xi)+b_{-1}(x,\xi)+\ldots+b_{-k}(x,\xi)$ then $P_1\circ P_2$ has symbol $c=c_0(x,\xi)+c_{-1}(x,\xi)+\ldots+c_{-k}(x,\xi)$ where

$$\begin{aligned} c_0 &= a_0b_0 \\ c_{-1} &= a_0b_{-1}+a_{-1}b_0+\sum_{i=1}^{n}\partial_{\xi_i}a_0\partial_{\xi_i}b_0 \\ c_{-2} &= a_{-1}b_{-1}+a_0b_{-2}+a_{-2}b_0+\sum_{i,j}\partial_{x_i}\partial_{x_j}a_0\partial_{\xi_i}\partial_{\xi_j}b_0 \\ &\quad+\sum_{i=1}^{n}\partial_{x_i}a_0\partial_{\xi_i}b_{-1}+\sum_{i=1}^{n}\partial_{x_i}a_{-1}\partial_{\xi_i}b_0 \\ &\text{etc.} \end{aligned} \tag{5.5}$$

By Proposition 5.2 it is clear that the H^{s+k-j} norm of c_{-j} is bounded by $C\|P_1\|_s\|P_2\|_s$, for example

$$\begin{aligned} \|c_0\|_{s+k} &\leqq \|a_0\|_{s+k}\|b_0\|_{s+k} \\ \|c_{-1}\|_{s+k-1} &\leqq \|a_0\|_{s+k-1}\|b_{-1}\|_{s+k-1}+\|a_{-1}\|_{s+k-1}\|b_0\|_{s+k-1} \\ &\quad+n\|a_0\|_{s+k}\|b_0\|_{s+k} \\ &\text{etc.}\quad \square \end{aligned} \tag{5.6}$$

Thus $\psi DO^s_{0,k}$ is a Hilbert algebra. It follows that the subset $(\psi DO^s_{0,k})_*$ of invertible elements in $\psi DO^s_{0,k}$ is open and that inversion, $P\mapsto P^{-1}$, is a homeomorphism of $(\psi DO^s_{0,k})_*$. If P has symbol $a=a_0(x,\xi)+a_{-1}(x,\xi)+\ldots+a_{-k}(x,\xi)$ then P^{-1} has symbol $\tilde{a}=\tilde{a}_0(x,\xi)+\ldots+\tilde{a}_{-k}(x,\xi)$ where $\tilde{a}_0(x,\xi)=\dfrac{1}{a_0(x,\xi)}$ and $\tilde{a}_{-j}(x,\xi)$ depends only on $a_{-j}(x,\xi)$, the first derivatives of $a_{-j+1}(x,\xi)$, the second derivatives of $a_{-j+2}(x,\xi)$, and so on. Composition in $(\psi DO^s_{0,k})_*$ is in fact smooth since it is the restriction of the composition map in $\psi DO^s_{0,k}$ which is bilinear and continuous, hence smooth. Therefore $(\psi DO^s_{0,k})_*$ is a Hilbert Lie group.

Summarizing we have the following

Theorem 5.6. $\{(\psi DO_{0,k})_*, (\psi DO^s_{0,k})_* | s>\dim M\}$ *is an* ILH-*Lie group where each* $(\psi DO^s_{0,k})_*$ *is a smooth Hilbert Lie group. Its* ILH-*Lie algebra is* $\{\psi DO_{0,k}, \psi DO^s_{0,k} | s>\dim M\}$.

Remark. Now that we have given $(\psi DO_{0,k})_*$ an ILH-Lie group structure we can take limits to describe a Lie group structure for $(\psi DO)_*$. First, consider $(\psi DO_0)_*$:

Corollary 5.7. $(\psi DO_0)_*$ *has the Lie group structure of a direct limit of* ILH-*Lie groups, i.e.* $(\psi DO_0)_*=\varinjlim(\psi DO_{0,k})_*$.

This situation for the full group ψDO_* is somewhat more complicated. For any m we can give $\psi DO_m \cap \psi DO_*$ the structure of a direct limit of ILH-manifolds by using an elliptic operator to identify $\psi DO_m \cap \psi DO_*$ with $(\psi DO_0)_*$. Multiplication will be smooth in the appropriate sense between the appropriate spaces. Piecing this together for all m makes ψDO_* a *graded direct limit of ILH-Lie groups*. We omit precise formulations of these concepts since they are direct generalizations of Sect. 3.

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